Matrix Revision

Vera Mann

19 July 2002

Recommended reading:
1) For those with little experience of matrices; handout drawn from Essential Pure Mathematics (Backhouse), pp. 229-245
3) Any 'Matrix algebra' book or 'Matrix algebra summary' in statistics books
4) Useful web sites can be found searching for "matrix algebra" at www.google.uk.co.

Basic definitions and notations:

An \( m \times n \) matrix \( A \) (or in some books \( A \) ) is a rectangular array of numbers with \( m \) rows and \( n \) columns

\[
A = \begin{pmatrix}
    a_{11} & \cdots & a_{1n} \\
    \vdots & \ddots & \vdots \\
    a_{m1} & \cdots & a_{mn}
\end{pmatrix} = \{a_{ij}\}, \quad i = 1, \cdots, m \text{ (rows) and } j = 1, \cdots, n \text{ (columns)}
\]

The elements of a matrix \( A(m \times n) \) are \( a_{ij} \)

The order of a matrix is the number of rows by the number of columns i.e. \( m \times n \)

A column vector with \( m \) elements, \( y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} \), is a matrix with only one column i.e. an \( m \times 1 \) matrix

A row vector with \( n \) elements, \( x = \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \), is a matrix with only one row i.e. an \( 1 \times n \) matrix
**Transposed matrix** $A^T$ (or $A'$) arises from the matrix $A$ by interchanging the column vectors and the row vectors i.e. $a_{ij}^T = a_{ji}$ (so a column vector is converted into a row vector and visa versa)

A partitioned matrix is a matrix written in terms of sub-matrices

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11}, A_{12}, A_{21}, A_{22}$ are sub-matrices.

Notes:

1. $A_{11}$ and $A_{21}$ have the same number of columns, so do $A_{12}$ and $A_{22}$
2. $A_{11}$ and $A_{12}$ have the same number of rows, so do $A_{21}$ and $A_{22}$
3. partitioning is not restricted to dividing a matrix into just four sub-matrices.

A square matrix has exactly as many rows as it has columns i.e. the order of the matrix is $n \times n$

The main diagonal (or leading diagonal) of a square matrix $A(n \times n)$ are the elements lying on the diagonal from top left to bottom right

$a_{11}, a_{22}, \ldots, a_{nn}$ i.e. all $a_{ii} = 1, \ldots, n$

The trace of a square matrix is the sum of the diagonal elements

$$tr(A) = a_{11} + a_{22} + \ldots + a_{nn} = \sum_{i=1}^{n} a_{ii}$$

- **Examples:**

1. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is a $2 \times 3$ matrix, while $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ has order 3 by 2.

2. $x = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ is a $1 \times 3$ row vector and $y = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is a $2 \times 1$ column vector.

3. $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \end{pmatrix}$ can be written as $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, where

$A_{11} = \begin{pmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \end{pmatrix}$, $A_{12} = \begin{pmatrix} 4 & 5 \end{pmatrix}$, $A_{21} = \begin{pmatrix} 11 & 12 & 13 \end{pmatrix}$ and $A_{22} = \begin{pmatrix} 14 & 15 \end{pmatrix}$
4. \[ A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \] is a 2 \times 2 square matrix, the main diagonal of this matrix are the elements 1 and 4; and \( tr(A) = 1 + 4 = 5 \)

**Special matrices:**

A **symmetric matrix** is a square matrix for which the following is true for all the off diagonal elements

\[ a_{ij} = a_{ji} \quad i \neq j \quad \text{i.e.} \quad A^T = A. \quad \text{(i.e. it is symmetric about the diagonal)} \]

**Diagonal matrix** is a square matrix having zero for all the non-diagonal elements i.e.

\[ A = \begin{pmatrix} a_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{pmatrix}. \]

**Zero matrix** (null matrix) is a matrix whose all elements are zero

**Identity matrix** (or unit matrix) is a diagonal matrix having all diagonal elements equal to 1 and off diagonal elements equal to zero. i.e. \( I_{(n \times n)} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \)

"**Summing vector**" is a vector whose every element is 1 i.e. \( 1_{(n)} = \begin{pmatrix} 1 & \cdots & 1 \end{pmatrix} \)

"**J**" matrix is a matrix (not necessarily square) whose every element is 1 i.e. \( J_{(m \times n)} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \)

- **Examples:**

1. \[ A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix} \] is a symmetric matrix, while \[ A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \] is a diagonal matrix.
2. \( I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \) is a 3 \( \times \) 3 unit matrix, while \( J = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \) is a 3 \( \times \) 3 \( J \) matrix.

**Basic operations:**

**Addition (Substraction)** can take place only when the matrices involved are of the same order; i.e. two matrices can be added (subtracted) only if they have the same numbers of rows and the same numbers of columns.

\[
A \pm B = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \pm \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} \pm b_{11} & \cdots & a_{1n} \pm b_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} \pm b_{m1} & \cdots & a_{mn} \pm b_{mn} \end{pmatrix}
\]

- \( A + B = B + A \)
- \( (A + B) + C = A + (B + C) \)
- \( A + 0 = 0 + A = A \)
- \( A + (-A) = 0 \)
- \( (A + B)^T = A^T + B^T \)

**Multiplication by scalar** \( c \) means multiplying every element of the matrix by \( c \)

\[
cA = \begin{pmatrix} ca_{11} & \cdots & ca_{1n} \\ \vdots & \ddots & \vdots \\ ca_{m1} & \cdots & ca_{mn} \end{pmatrix}
\]

- \( cA = Ac \)
- \( c(dA) = (cd)A \)
- \( (c \pm d)A = cA \pm dA \)
- \( c(A \pm B) = cA \pm cB \)
Multiplication of an $2 \times 2$ matrix by a column vector which has 2 rows yields a column vector with 2 rows

\[ Ax = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{pmatrix} \]

generally:

Multiplication of an $m \times n$ matrix by a column vector which has $n$ rows yields a column vector with $m$ rows

\[ Ax = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \end{pmatrix} = y \]

i.e. $y_i = \sum_{j=1}^{n} a_{ij}x_j, \quad i = 1, \ldots, m$

**Multiplication of matrices** The product $AB = C$ is defined only when $A$ has exactly as many columns as $B$ has rows, and the elements of $C$ are given as

\[ c_{ij} = \sum_{l=1}^{n} a_{il}b_{lj}, \quad i = 1, \ldots, m \quad \text{and} \quad j = 1, \ldots, n \]

i.e. if $A$ is $m \times r$ matrix, $B$ is $r \times n$ matrix then $C$ will be $m \times n$ matrix

- $AB \neq BA$
- $(AB)C = A(BC) = ABC$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- $IA = AI = A$

**Examples:**

1. \[ \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{pmatrix} = \begin{pmatrix} 1 + 7 = 8 & 2 + 8 = 10 & 3 + 9 = 12 \\ 4 + 10 = 14 & 5 + 11 = 16 & 6 + 12 = 18 \end{pmatrix} \]
2. \[
\begin{pmatrix}
1 & 2 & 3 \\
4 & 5 & 6
\end{pmatrix}
\begin{pmatrix}
1 & 2 \\
3 & 4 \\
5 & 6
\end{pmatrix}
= \begin{pmatrix}
1 \times 1 + 2 \times 3 + 3 \times 5 & 1 \times 2 + 2 \times 4 + 3 \times 6 \\
4 \times 1 + 5 \times 3 + 6 \times 5 & 4 \times 2 + 5 \times 4 + 6 \times 6
\end{pmatrix}
= \begin{pmatrix}
22 & 28 \\
49 & 64
\end{pmatrix}
\]

3. \[
\begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix}
= \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
\]

4. \[
\begin{pmatrix}
2 & 3 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 1 \\
1 & 1
\end{pmatrix}
= \begin{pmatrix}
5 & 5 \\
5 & 5
\end{pmatrix}
\]

Further definitions:

The **determinant** of a second order square matrix is

\[
\text{det}(A) = |A| = \begin{vmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}
\]

Note: it is harder to define in general for matrices with higher order (see text books)

The **inverse** of a matrix \(A\), \(A^{-1}\) if it exists, is a matrix whose product with \(A\) is the identity matrix i.e. \(AA^{-1} = A^{-1}A = I\). (Note: both \(A\) and \(A^{-1}\) have to be square)

For second order matrices: \(A^{-1} = \frac{1}{\text{det}(A)} \begin{pmatrix}
a_{22} & -a_{12} \\
-a_{21} & a_{11}
\end{pmatrix}\)

**Singular** or non-invertible matrix: \(\text{det}(A) = 0\)

**Idempotent** matrices are square and the following is true:

\(AA = A^2 = A\)

**Orthogonal** matrices have the following property:

\(AA^T = A^TA = I\)
• Examples:

1. If \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \) then \( \text{det}(A) = \left| A \right| = 1 \times 4 - 2 \times 3 = -2 \)

   and the inverse is \( A^{-1} = -\frac{1}{2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \)

2. If \( A \) is a diagonal matrix with main diagonal elements of \( a_{ii} i = 1, \ldots, n \) then \( A^{-1} \) is also diagonal with main diagonal elements of \( 1/a_{ii} i = 1, \ldots, n \)

\[
\begin{pmatrix} a_{11} & \ldots & 0 \\
0 & \ddots & \vdots \\
0 & \ldots & a_{22} \end{pmatrix} \begin{pmatrix} 1/a_{11} & \ldots & 0 \\
0 & \ddots & \vdots \\
0 & \ldots & 1/a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \cdot 1/a_{11} & \ldots & 0 \\
0 & \ddots & \vdots \\
0 & \ldots & a_{22} \cdot 1/a_{22} \end{pmatrix} = I
\]

3. \( A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \) is an idempotent matrix

\[
\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} = \begin{pmatrix} 1/4 + 1/4 & 1/4 + 1/4 \\ 1/4 + 1/4 & 1/4 + 1/4 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}
\]